Optimization of Subsonic and Transonic Airfoils for Natural Laminar Flow using a Discrete-Adjoint Method

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The design of natural-laminar-flow airfoils is demonstrated by high-fidelity, multipoint, aerodynamic shape optimization capable of efficiently incorporating and exploiting laminar-turbulent transition. First, a two-dimensional Reynolds-averaged Navier-Stokes (RANS) flow solver has been extended to incorporate an iterative laminar-turbulent transition prediction methodology. The natural transition locations due to Tollmien-Schlichting instabilities are predicted using the simplified $e^N$ envelope method of Drela and Giles or alternatively, the compressible form of the Arnal-Habiballah-Delcourt criterion. The boundary-layer properties are obtained directly from the Navier-Stokes flow solution, and the transition to turbulent flow is modeled using an intermittency function in conjunction with the Spalart-Allmaras turbulence model. The RANS solver is subsequently employed in a gradient-based sequential quadratic programming shape optimization framework. The laminar-turbulent transition criteria are tightly coupled into the objective and gradient evaluations. The gradients are obtained using a new augmented discrete-adjoint formulation for non-local transition criteria. The aerodynamic design requirements are cast into a multipoint design optimization problem. A composite objective is defined using a weighted integral of the operating points. The proposed framework is applied to the single and multipoint optimization of subsonic and transonic airfoils, leading to robust natural-laminar-flow designs.

I. Introduction and Motivation

The current push for environmentally responsible aviation requires serious efforts to mitigate the escalating effects of such technology on climate change and natural resources. A clear vision for the efficiency of future transport aircraft – with specific targets for reduced fuel burn, emissions and noise – has been published in the U.S. National Aeronautics Research and Development Plan.1 As part of the effort to reduce fuel burn and emissions, aerodynamicists are assessing the feasibility of natural laminar flow (NLF) as a key enabler of environmentally responsible commercial aviation.

Over the past few decades, the use of CFD under the assumption of fully-turbulent conditions has allowed for significant advancements in aerodynamic design, but the conservatism leaves something to be desired. Indeed, design tools capable of incorporating and exploiting laminar-turbulent transition enable the design of aircraft with significantly reduced drag. The lack of NLF applications in the fleet points to the sparsity of available design tools for NLF; it also points to the challenges in reliably realizing extended regions of laminar flow in flight. The transition to turbulence is affected by many factors, including: Reynolds number ($Re$), freestream turbulence intensity ($T_u$), pressure gradient, Mach number ($M$), surface roughness and heating, structural noise, rain, hail, icing, and insect impacts.2,3,4,5

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Research in the area of high-fidelity aerodynamic shape optimization with laminar-turbulent transition is sparse. The majority of research in this field employs inviscid-viscous coupling strategies, making use of boundary-layer codes for the viscous formulation and either a panel method or the Euler equations for the inviscid formulation. Several researchers have attempted to optimize for NLF using RANS solvers with algorithms that include both local and non-local transition prediction criteria, as well as gradient and gradient-free optimization strategies.

In this work, NLF design is demonstrated through high-fidelity, single and multipoint aerodynamic shape optimization with transition prediction capable of accounting for the effects of $Re$, $Tu$, $M$, and pressure gradient. The Reynolds-averaged Navier-Stokes (RANS) equations are solved with the one-equation Spalart-Allmaras (SA) turbulence model. The solver is first extended to incorporate an iterative laminar-turbulent transition prediction methodology, and is subsequently employed in a gradient-based Sequential Quadratic Programming (SQP) shape optimization framework. Practical design requirements are cast into a parallelized multipoint design optimization problem. The proposed framework presents a good compromise between accuracy, robustness, and efficiency, resulting in a flexible and high-fidelity RANS-based optimization framework for NLF design in subsonic and transonic flight.

### II. Flow Solver Methodology

The steady RANS equations are solved in two dimensions using a second-order Newton-Krylov finite-difference flow solver (named Optima2D) originally developed by Nemec and Zingg. The linear system that arises at each Newton iteration is solved using the preconditioned Generalized Minimum Residual (GMRES) method. Global convergence of the Newton method is made possible by an approximate factorization startup algorithm. Numerical dissipation is added by either the scalar dissipation scheme of Jameson et al. or the matrix dissipation scheme of Swanson and Turkel. The turbulent eddy viscosity is computed using the one-equation Spalart-Allmaras (SA) turbulence model. The SA model is not itself capable of predicting transition; the remaining constituents of the proposed transition prediction framework include: the determination of the boundary-layer edge and properties, the calculation and evaluation of the transition criteria, and the implementation of a robust iterative procedure for transition prediction in the RANS solver.

Three boundary-layer edge-finding methods have been implemented, verified and compared in the RANS flow solver; details may be in Rashad and Zingg. The accuracy of the integrated boundary-layer properties has also been assessed through a detailed grid convergence study and by comparison to numerical boundary-layer properties obtained from XFOIL. It was found that with reasonable grid density, the boundary-layer properties can be computed directly from the Navier-Stokes solution with sufficient accuracy. The remainder of this section presents the transition criteria and discusses their implementation in the RANS solver.

The natural transition locations (due to Tollmien-Schlichting instabilities) are predicted using the simplified $e^N$ envelope method used in Drela’s XFOIL and MSES codes. The method makes direct use of the boundary-layer properties to approximate the envelope of the spatial amplification rates of the disturbances (the N-factors), as opposed to actually solving the linear stability equations. While the envelope method does not track individual frequencies, it is significantly more efficient. The new compressible form of the Arnal-Habiballah-Delcourt (AHD) criterion has also been incorporated into the RANS solver. The AHD criterion is designed for low to moderate freestream turbulence intensities ($Tu \leq 0.1\%$), as typically encountered in external aerodynamic cruise conditions for transport aircraft. The details of these criteria are omitted herein for brevity.
II.A. RANS Implementation

II.A.1. Iterative Transition Prediction Procedure

Automatic transition prediction in the RANS solver is achieved through an iterative process, which has been developed by several researchers.\textsuperscript{5,33,29,28,34,32} This section provides an overview of the current implementation.

An initial guess of the transition locations (top and bottom surfaces) is required and is typically taken at 25\% chord. Transition is then forced to occur at the initial locations using a transition region model (Section II.A.2). When the magnitude of the flow residual has been reduced to $5 \times 10^{-6}$, the transition prediction module is invoked to process the RANS solution; the tight tolerance was chosen to ensure sufficiently accurate boundary-layer properties for transition prediction. The forced transition points are then moved upstream or downstream as required toward the predicted transition points in an under-relaxed fashion,\textsuperscript{34} such that

$$X_{t}^{\text{new}} = X_{t}^{\text{old}} - \omega \left( X_{t}^{\text{old}} - X_{p} \right),$$  \hspace{1cm} (1)

where $\omega$ is the under-relaxation factor, and $X_{t}$ and $X_{p}$ represent the normalized chord locations of the forced and predicted transition points, respectively. When the flow residual returns to a magnitude of $5 \times 10^{-6}$, the predicted and forced transition points are again updated. The iterative transition prediction procedure is considered converged when the absolute value of the transition residual, $|R_{t}| = |X_{t} - X_{p}|$, has converged to a tolerance of $\epsilon_{tp}$. For the purposes of gradient-based aerodynamic shape optimization, $\epsilon_{tp}$ and $\epsilon_{r}$ are set to $10^{-8}$ and $10^{-12}$, respectively, ensuring a sufficiently smooth design space for optimization.

From numerical experimentation, an under-relaxation factor of $\omega = 0.8$ has been selected as a good compromise between efficiency and robustness. A linear extrapolation of the boundary-layer properties – from the laminar region into the turbulent region – allows the transition criterion to predict transition downstream of the forced transition points (when required). If laminar flow separation is detected and a transitional separation bubble forms, the $e^{N}$ envelope method is able to predict the transition location in the separation bubble.\textsuperscript{27} When using the AHD criterion, the laminar separation point is simply taken as the transition point. A robust logic has been determined through extensive numerical experimentation and code verification to handle the various outcomes of the transition prediction module. For the various airfoils and flight conditions investigated, it was found that the iterative transition prediction procedure requires approximately three to four times the computational cost of a fully-turbulent flow solve, with no significant addition to the memory requirements.

II.A.2. Modelling of Transitional Flow Regions

The transition to turbulence is enforced in the Navier-Stokes solution by one of two methods. The first makes use of the trip term and the $f_{t1}$ and $f_{t2}$ trip functions in the SA model, as published by Spalart and Allmaras.\textsuperscript{26} The second approach makes use of an intermittency function that scales the turbulent eddy viscosity, such that $\mu_{t} = \gamma \mu_{t}$ and $0 \leq \gamma \leq 1$, as used by Cliquet et al.\textsuperscript{29} The intermittency function has been defined to take the form of an S-type curve with a smooth initial ramp-up, such that

$$\gamma(x) = \exp(-5 \xi^{2}), \quad \text{where} \quad \xi = 1 + \frac{x_{\text{beg}}^{\text{tr}} - x}{l_{\text{tr}}},$$  \hspace{1cm} (2)

$x_{\text{beg}}^{\text{tr}}$ represents the beginning of the transitional flow region as predicted by the transition criterion, and $l_{\text{tr}}$ is the transition length. Although there are no physics-based methods for determining the transition length,\textsuperscript{28} empirically correlated approximations have been developed that make use of the boundary-layer properties
at the transition point. Following the work of Krumbein, the transition length can be obtained from

\[ Re_{tr} = 4.6 \left( Re^{*}_{tr} \right)^{1.5}. \]

For a smooth ramp-up of the eddy viscosity, the transition region must be sufficiently resolved; failure to do so was observed to cause noise in the design space during optimization. A comparison of the eddy viscosity ramp-up using the intermittency function as compared to the Spalart-Allmaras trip terms may be found in Rashad and Zingg.

III. Optimization Framework

The goal of the aerodynamic shape optimization framework is to minimize the specified design objective, \( J \), with respect to the design variables, \( X \), subject to linear and nonlinear constraints. Although the optimizer can handle several different design objectives, such as the maximization of lift-to-drag ratio or endurance factor, in this work the focus is on lift-constrained drag minimization. The proposed optimization framework consists of the following: (i) a two-dimensional RANS flow solver (described in the preceding section), (ii) a geometry parametrization and mesh movement algorithm, (iii) a sequential quadratic programming algorithm, and (iv) a discrete-adjoint gradient computation.

The airfoil geometry is parametrized using B-splines, the details of which may be found in Nemec and Zingg. The design variables, \( X \), are defined as the \( y \)-coordinates of the B-spline control points; the control points are free to move in the vertical direction to facilitate shape changes during the optimization cycle. The angle of attack of the airfoil is an additional design variable. The algebraic grid-perturbation strategy described in Nemec and Zingg is used to ensure that the computational grid is smoothly adjusted to conform to the changing geometric configurations.

The SNOPT general purpose Sequential Quadratic Programming (SQP) algorithm – developed by Gill et al. – is employed as the optimizer in this work. SNOPT requires the gradients of the objective function and constraints; ensuring sufficiently accurate gradients is of paramount importance to the success of the SQP algorithm. Two methods for computing accurate gradients (that incorporate the sensitivities of the transition criterion) have been implemented: a parallel finite-difference gradient evaluation, along with a new augmented discrete-adjoint gradient evaluation, presented in the next section.

III.A. Discrete-Adjoint Gradient Evaluation

The principal advantage of the adjoint method is that its cost does not scale with the number of design variables, but rather with the number of objectives and nonlinear constraints. Hence, the objective function gradient evaluation only requires one flow solve and one adjoint solve; for lift-constrained drag minimizations, an additional adjoint solve is required for the gradient of the lift-constraint. A detailed description and derivation of the discrete-adjoint formulation in the context of aerodynamic shape optimization may be found in Nemec and Zingg.

In the discrete-adjoint approach, the gradient is evaluated using the following expression:

\[ G = \frac{dJ}{dX} = \frac{\partial J}{\partial X} - \psi^{T} \frac{\partial R}{\partial X}, \]

where \( R = R[X, Q(X)] \) represents the discretized RANS residual vector. The vector of adjoint variables, \( \psi \), is obtained by solving the linear system of equations given by

\[ \frac{\partial R^{T}}{\partial Q} \psi = \frac{\partial J}{\partial Q}, \]

where \( Q \) is the vector of conserved flow variables.
III.A.1. Adjoint Formulation for Transition Prediction

The AHD and \( e^N \) transition criteria are non-local in their formulation. As such, special consideration must be taken when evaluating and deriving an adjoint formulation capable of incorporating their sensitivities. The proposed approach is to append a new adjoint vector, \( \psi_{tr} \), to the original adjoint vector, such that \( \overline{\psi} = [\psi ; \psi_{tr}] \). Henceforth, the overbar shall be used to indicate an augmented vector. The length of \( \psi_{tr} \) corresponds to the number of transition points, \( N_{tr} \), which is equal to two for a single-element airfoil. For 3D wing configurations, the transition lines may be defined by a spanwise distribution of transition points; the total number of spanwise transition points on all surfaces gives \( N_{tr} \).

To compute the new adjoint variables, we specify a corresponding number of new residual equations, such that \( \overline{R} \rightarrow [R ; R_{tr}] \). The new transition residual equations represent the distance between the forced and predicted transition locations, \( R_{tr} = X_f - X_p \), described in Section II.A.1. The transition residual vector is satisfied \( (R_{tr} = 0) \) when the forced transition points are in locations consistent with the given transition criterion \( (X_f = X_p) \).

In addition, the vector of conserved flow variables must be augmented to include the forced transition locations, such that \( \overline{Q} \Rightarrow [Q ; X_f] \). Finally, the entire adjoint vector, \( \overline{\psi} \), is computed by solving the augmented linear system of equations given by

\[
\frac{\partial \overline{R}}{\partial \overline{Q}}^T \overline{\psi} = \frac{\partial J}{\partial \overline{Q}}^T \quad \text{subject to} \quad \overline{R}_{tr} \overline{\psi}_{tr} = 0 \quad \text{(6)}
\]

The \( \frac{\partial R}{\partial X_f} \) matrix represents the sensitivity of the flow residual to the forced transition points. It is computed efficiently using a centered difference approximation requiring only two evaluations of the flow residual for each transition point. The matrix \( \frac{\partial R_{tr}}{\partial X_f} \) represents the sensitivity of the transition residual to the forced transition points, which by the definition of \( R_{tr} = X_f - X_p \), is simply the \( N_{tr} \times N_{tr} \) identity matrix. Furthermore, the vector \( \frac{\partial J}{\partial X_f} \) is simply the null vector for typical objectives and constraints such as lift and drag, since these objectives do not depend explicitly (but rather implicitly) on the transition points. The matrix \( \frac{\partial R_{tr}}{\partial Q} \) is by far the most complex of the new matrices in the augmented formulation as it represents the sensitivity of the transition residual (including the evaluation of the boundary-layer edge, the boundary-layer properties, and the given transition criterion) to the conserved flow variables. This matrix is computed accurately using a complex-step approximation.37, 38

III.A.2. Solving the Augmented Adjoint System

An iterative block-based approach is proposed to solve the augmented adjoint system. The approach makes use of a Generalized Minimum Residual (GMRES) Krylov subspace solver in an iterative fashion. The solution procedure requires an initial guess for the transition adjoint variables, taken as \( \psi_{tr}^{n=1} = 0 \), and is summarized as follows:

1. Use preconditioned GMRES to solve the following linear system of equations for \( \psi_{tr}^{n+1} \), where \( n \) is the iteration counter:

\[
\frac{\partial R}{\partial Q}^T \psi_{tr}^{n+1} = \frac{\partial J}{\partial Q}^T - \frac{\partial R_{tr}}{\partial Q}^T \psi_{tr}^{n} \quad \text{subject to} \quad \frac{\partial R_{tr}}{\partial Q}^T \psi_{tr}^{n+1} = 0 \quad \text{(7)}
\]
2. Update the $\psi_{tr}$ vector by solving the following linear system of equations (directly):

$$\frac{\partial R_{tr}}{\partial X_f}^{T} \psi_{tr}^{n+1} = \frac{\partial J}{\partial X_f}^{T} - \frac{\partial R}{\partial X_f}^{T} \psi_{n+1}.$$  

(8)

Note that since $\frac{\partial R_{tr}}{\partial X_f}$ is the identity matrix and $\frac{\partial J}{\partial X_f}$ is the null vector, (8) simplifies to

$$\psi_{tr}^{n+1} = -\frac{\partial R}{\partial X_f}^{T} \psi_{n+1}.$$  

(9)

3. Increment the iteration counter, $n \leftarrow n + 1$, and return to step 1.

4. Stop when a given convergence criterion is satisfied. The convergence criterion selected for the iterative procedure makes use of the L2-norm of the augmented adjoint system, such that

$$\left\| \frac{\partial R}{\partial Q}^{T} \psi - \frac{\partial J}{\partial Q}^{T} \right\|_2 \leq \epsilon_{adj}.$$  

(10)

A value of $\epsilon_{adj} = 10^{-8}$ has been selected to ensure that the final solution vector, $\overline{\psi}$, satisfies the augmented system of equations sufficiently well.

One of the main advantages of the proposed iterative approach is that, in step 1, the same first-order preconditioner used to precondition GMRES during the flow solution (and the original adjoint solution) can also be used here to precondition the system given by (7). Furthermore, in the above algorithm GMRES makes use of the same left-hand side during each iteration, with only the right-hand side being modified. It follows that the robustness of GMRES that has been observed in fully-turbulent and fixed transition flow solutions is preserved here. Another advantage is that the method is easy to implement since it uses the same matrix-vector products in step 1 as the fully-turbulent and fixed transition solutions, and it requires no modifications within the GMRES solver itself. A principal disadvantage of the iterative approach is the time required to solve (7) multiple times. However, with the appropriate selection of an initial guess and by the under-relaxation of the updates, the method is relatively quick to converge, typically requiring between 5 and 10 iterations. For example, it has been observed that the time required to compute the gradients for a lift-constrained drag minimization (both $\partial C_d/\partial X$ and $\partial C_l/\partial X$) using the proposed algorithm is equal to or less than the time required to compute a single flow solution with free transition.

III.B. Multipoint Optimization

We use multipoint optimization to ensure that our aerodynamic designs perform reasonably well over a given flight envelope. This is particularly important in the design of NLF airfoils, which, in order to maximize the extent of laminar flow, tend to take the boundary-layer very close to the point of separation prior to pressure recovery. We use the methodology of Buckley and Zingg\textsuperscript{39, 40} to perform multipoint optimization capable of handling a comprehensive set of aerodynamic design requirements. In particular, we are interested in considering a range of Reynolds numbers, Mach numbers, and aircraft weights ($W$). The optimizer then minimizes the weighted integral of the objective (in our case, the drag coefficient) subject to the corresponding lift constraints (one for each design point). Also note that each operating point has an associated angle of attack, all of which are included as additional design variables. The weighted integral is approximated as follows:\textsuperscript{40}

$$J = \sum_{i=1}^{N_W} \sum_{j=1}^{N_M} T_{i,j} C_d(M_i, W_j) \Delta M \Delta W,$$  

(11)
Table 1. Optimization cases

<table>
<thead>
<tr>
<th>Case</th>
<th>Aircraft</th>
<th>Reynolds Number ((Re))</th>
<th>Mach Number ((M))</th>
<th>Lift Coefficient ((C_l^*))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Cessna 172R</td>
<td>5.6 \times 10^6</td>
<td>0.19</td>
<td>0.30</td>
</tr>
<tr>
<td>B</td>
<td>Dash-8 Q400</td>
<td>15.7 \times 10^6</td>
<td>0.60</td>
<td>0.42</td>
</tr>
<tr>
<td>C</td>
<td>Boeing 737-8</td>
<td>20.3 \times 10^6</td>
<td>0.71</td>
<td>0.50</td>
</tr>
</tbody>
</table>

where \(N_M\) and \(N_W\) are the numbers of quadrature points, and \(\Delta M\) and \(\Delta W\) are the corresponding spacings between quadrature points. The \(T_{i,j}\) are the associated quadrature weights used to approximate the integral. In this work, the trapezoidal quadrature rule is employed. Finally, \(Z\) is a weighting function to be specified by the designer. This weighting function allows the designer to specify the importance of each design point according to their own priorities.

The above multipoint formulation requires one flow solution and two adjoint solutions for each operating point. Buckley and Zingg\(^40\) have parallelized the multipoint framework such that multiple processors compute the necessary objective, constraint, and gradient information. This approach has been shown to be an effective technique for robust and efficient aerodynamic design over a range of operating conditions.\(^39, 40\)

Full details of the various operating conditions and their associated weights are presented in Section IV.F.

IV. Results

IV.A. Optimization Results - Problem Definitions

To demonstrate the NLF design capabilities of the optimization framework, single and multipoint optimizations are performed at conditions associated with subsonic and transonic commercial aircraft. The verification and validation of the flow solver’s transition prediction capabilities may be found in Rashad and Zingg.\(^17\) The design objective is to minimize the total drag of the airfoil constrained by a user-specified lift target, \(C_l^*\). For structural considerations, additional inequality constraints are included. An area constraint ensures that the final area of the airfoil is greater than or equal to the initial area, and a thickness constraint near the leading edge ensures a minimum thickness of 2.5% chord located at 2.7% chord.

Single-point optimization results for all of the cases outlined in Table 1 are presented. Cases A, B, and C were selected to approximate the cruise flight conditions of the Cessna 172R, the Bombardier Dash-8 Q400 turboprop, and the Boeing 737-800 turbofan, respectively. Multipoint optimization is performed on the Dash-8 Q400 and is presented in section IV.F.

The initial geometry for all cases is the RAE-2822 airfoil parametrized by seventeen B-spline control points. The control point located at the leading edge of the airfoil, as well as the two coincident control points at the trailing edge, are kept fixed throughout the optimization. The \(y\)-coordinates of the remaining 14 control points are used as the geometric design variables. The angle of attack is also included as an additional design variable. The computational grid consists of a \(575 \times 224\) \(C\)-grid, resulting from grid convergence studies on the boundary-layer properties, with flow solutions computed using the scalar dissipation scheme of Jameson \textit{et al.}\(^24\) All results were obtained using the compressible Bernoulli edge-finding method, the intermittency function transition region model, the discrete-adjoint based gradient evaluation, and the \(e^N\) transition criterion. Prior to discussing the optimization results, the next section presents an accuracy assessment of the discrete-adjoint gradient evaluation.
Table 2. Comparison of finite-difference (FD) and discrete-adjoint (AD) gradient components

<table>
<thead>
<tr>
<th>Component</th>
<th>FD</th>
<th>AD</th>
<th>Diff. (AD−FD)</th>
<th>% Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0679450100</td>
<td>0.0679451512</td>
<td>1.4123964E-07</td>
<td>0.0002078735</td>
</tr>
<tr>
<td>2</td>
<td>0.0219869160</td>
<td>0.0219869646</td>
<td>4.8564238E-08</td>
<td>0.0002208779</td>
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<tr>
<td>3</td>
<td>0.0097466881</td>
<td>0.0097467000</td>
<td>-1.8115582E-08</td>
<td>-0.0001858640</td>
</tr>
<tr>
<td>4</td>
<td>0.0175834070</td>
<td>0.0175834517</td>
<td>4.4744770E-08</td>
<td>0.0002544716</td>
</tr>
<tr>
<td>5</td>
<td>-0.0193916140</td>
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<tr>
<td>6</td>
<td>-0.0299716320</td>
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<tr>
<td>7</td>
<td>-0.0047241138</td>
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<td>-0.0018237235</td>
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<tr>
<td>8</td>
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<td>0.2247165247</td>
<td>9.1469697E-07</td>
<td>0.0004070465</td>
</tr>
<tr>
<td>9</td>
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<td>10</td>
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<td>0.0423509102</td>
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<td>AoA</td>
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<td>0.0010081334</td>
<td>1.2597056E-08</td>
<td>0.0012495582</td>
</tr>
</tbody>
</table>

IV.B. Verification of Discrete-Adjoint Gradient

In order to verify the accuracy of the augmented adjoint formulation, we compare the resulting gradient vector to a finite-difference gradient vector. If the correct step size is selected, then each component of the gradient vectors should be equal to within a small tolerance. For this accuracy assessment we perform a single iteration of the optimization to obtain the discrete-adjoint gradient (that is, the sensitivity of the drag coefficient to the design variables) for the initial geometry and parameterization described in Section IV.A, under the flight conditions of Case B. We also perform a centered-difference approximation by perturbing each design variable individually using a step-size of $1 \times 10^{-6}$. Table 2 compares the resulting adjoint (AD) and finite-difference (FD) gradient vectors for all design variables; the first 14 components are the geometric design variables, the last is the angle of attack. The results demonstrate excellent agreement between the two methods for computing the gradient, with the finite-difference gradient requiring 30 flow solutions, compared to a single flow solution (and a single adjoint solution) required for the adjoint gradient. The gradients of the nonlinear lift constraint show slightly better agreement and are omitted for brevity. Further gradient accuracy verifications have been carried out at different flight conditions and geometries, with similar results. Having verified the feasibility of the augmented adjoint formulation for transition prediction and the accuracy of its resulting gradient, the remainder of this paper is devoted to the presentation of the single and multipoint optimization results obtained using the discrete-adjoint gradient.

IV.C. Case A Results: $Re = 5.6 \times 10^6$, $M = 0.19$, $C_l^* = 0.3$

The Cessna 172R is assumed to be cruising at 6000 ft, a speed of 120 knots and a weight of 2200 lbs. The results were obtained using the $e^N$ envelope transition criterion with $N_{crit} = 9$. In Table 3, a summary of the results comparing the initial and optimized airfoils is presented. Figure 1(a) compares the initial and optimized geometries and Figure 1(b) compares the pressure profiles. The transition locations are indicated by the solid circles. The angle of attack increased from an initial value of 0.69° to 0.93°, the lift constraint is satisfied, and the total drag is reduced by 21.8 drag counts, or 45%. The ability of the optimizer to exploit
Table 3. Case A summary of optimization results: \( Re = 5.6 \times 10^6, M = 0.19, C^*_l = 0.3 \)

<table>
<thead>
<tr>
<th></th>
<th>( C_d )</th>
<th>( C_{d_p} )</th>
<th>( C_{d_v} )</th>
<th>( C_l )</th>
<th>( C_m )</th>
<th>( T_{up}(x/c) )</th>
<th>( T_{lo}(x/c) )</th>
<th>( AoA )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0.00482</td>
<td>0.00083</td>
<td>0.00399</td>
<td>0.3000</td>
<td>-0.06797</td>
<td>0.4918</td>
<td>0.5371</td>
<td>0.6858°</td>
</tr>
<tr>
<td>Optimized</td>
<td>0.00264</td>
<td>0.00072</td>
<td>0.00191</td>
<td>0.2993</td>
<td>-0.05294</td>
<td>0.8050</td>
<td>0.8500</td>
<td>0.9250°</td>
</tr>
</tbody>
</table>

Figure 1. Case A optimization results: \( Re = 5.6 \times 10^6, M = 0.19, C^*_l = 0.3 \); symbols indicate transition point locations.

The laminar-turbulent transition prediction is made evident by the aft movement of the transition points from 49\% to 81\% chord on the upper surface and 53\% to 85\% chord on the lower surface. The leading edge radius has decreased, and the point of maximum thickness has been pushed significantly aft in order to extend the region of favourable pressure gradient.

IV.D. Case B Results: \( Re = 15.7 \times 10^6, M = 0.60, C^*_l = 0.42 \)

The design point for the Dash-8 Q400 is taken as point 6 from the multipoint optimization case (discussed in Section IV.F). The results are obtained using the \( e^N \) envelope transition criterion with \( N_{crit} = 9 \). Table 4 provides a summary of the results comparing the initial and optimized airfoils. In this case, the angle of attack is decreased from an initial value of 1.14° to 0.80°, the lift constraint is again satisfied, and the total drag is reduced by 31.5 drag counts, or 53\%. The transition point on the upper surface has been moved aft by over 50\% chord, while the lower surface transition point has moved aft approximately 20\% chord.

Figure 2(a) compares the initial and optimized geometries; Figure 2(b) compares the pressure profiles. It can be observed that the optimizer was again successful in designing an airfoil with an extended favourable pressure gradient on both the upper and lower surfaces. As in the previous case, the optimized geometry has a smaller leading edge radius, and the location of maximum thickness has been moved aft. These results demonstrate the ability of the optimizer to design new NLF airfoils which would typically require considerable aerodynamic experience to design.
Table 4. Case B summary of optimization results: $Re = 15.7 \times 10^6$, $M = 0.60, C_l^* = 0.42$

<table>
<thead>
<tr>
<th></th>
<th>$C_d$</th>
<th>$C_{dp}$</th>
<th>$C_{dv}$</th>
<th>$C_l$</th>
<th>$C_m$</th>
<th>$T_{up}(x/c)$</th>
<th>$T_{lo}(x/c)$</th>
<th>AoA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0.00591</td>
<td>0.00189</td>
<td>0.00405</td>
<td>0.4200</td>
<td>-0.08129</td>
<td>0.1473</td>
<td>0.4911</td>
<td>1.1404°</td>
</tr>
<tr>
<td>Optimized</td>
<td>0.00276</td>
<td>0.00090</td>
<td>0.00187</td>
<td>0.4198</td>
<td>-0.07545</td>
<td>0.6934</td>
<td>0.7507</td>
<td>0.8044°</td>
</tr>
</tbody>
</table>

Figure 2. Case B optimization results; $Re = 15.7 \times 10^6$, $M = 0.60, C_l^* = 0.42$; symbols indicate transition point locations

IV.E. Case C Results: $Re = 20.3 \times 10^6$, $M = 0.71, C_l^* = 0.50$

The Boeing 737-800 has a wing sweep angle of 25° and is assumed to be cruising at 35000 ft and a Mach number of 0.785, which corresponds to an effective Mach number of 0.71. The target lift coefficient is 0.5. Results are obtained using the $e^N$ envelope transition criterion with $N_{crit} = 9$. Due to the transonic flight conditions, the optimization in this case is less robust. The flow solver may fail to converge if the transition locations are moved aft of a shockwave during the transition prediction procedure, in turn causing unsteady flow separation. Modification to the transition prediction algorithm includes a more gradual movement of the transition locations, restricted to a maximum of 5% chord at each update. It is also recommended to start with initial transition locations that are well upstream of any potential shock waves; an initial guess of 25% chord is used here for both the upper and lower surfaces.

Table 5 provides a summary of the results comparing the initial and optimized airfoils. The angle of attack in this case is decreased from an initial value of 1.13° to 0.15°, the lift constraint is satisfied, and the total drag is reduced by 29 drag counts, or 47%. The transition points are moved from 20% to 70% chord on the upper surface, and from 47% to 71% chord on the lower surface. Figure 3(a) compares the initial and optimized geometries; Figure 3(b) compares the pressure profiles. In this case, the optimizer is successful in designing a shock free natural laminar flow airfoil, in turn, significantly reducing the total drag.

IV.F. Case B: Multipoint Optimization

Here we consider a multipoint optimization at a range of cruise conditions associated with the Dash-8 Q400 aircraft. A nine-point stencil, presented in Table 6, is defined by varying the aircraft weight and
Mach number. This is done to reduce the sensitivity of the final optimized shape to variations in the flight conditions encountered during cruise and to enable efficient operation within this envelope. The aircraft is assumed to have a take-off weight equal to the Q400’s maximum take-off weight of 64500 lbs. Given a typical payload, the usable fuel on board (at take-off) is approximated to be 7500 lbs. The three aircraft weights considered in the multipoint stencil are calculated from a 10%, 50% and 90% fuel burn, which loosely approximates the beginning, middle, and end of cruise. The three Mach numbers considered are 0.6, 0.54, and 0.48, which roughly correspond to high-speed, intermediate, and long-range design speeds of the Q400, respectively. Assuming a constant cruising altitude of 23000 ft, we can then compute the corresponding range of Reynolds numbers and lift constraints presented in Table 6.

Although any design priority weighting, \( Z(W_i, M_j) \), may be selected as desired, here we make the assumption that all design points are of equal importance, that is, \( Z(W_i, M_j) = 1 \) for all \( i \) and \( j \). Table 6 also presents the quadrature weights \( T \) used in (11).

Table 7 provides a summary of the results comparing the initial and optimized airfoils, along with the various angles of attack. Note that the lift constraint has been satisfied and the drag reduced at each operating point. Figure 4(a) compares the initial and optimized geometries and Fig 4(b) compares the initial and optimized pressure distributions for design point 6. Note that the flight conditions and lift constraint of design point 6 corresponds to the Case B single-point optimization presented in Section IV.D. Comparing the optimized designs, it is clear that the single and multipoint results are different. The multipoint optimization has a transition point on the upper surface approximately 2.5% further upstream on the upper surface and 0.5% further downstream on the lower surface, as compared to the single-point optimization of Case B. This is due to the differences in the optimized geometries and angles of attack, which was increased from 0.8° in the single-point case to 1.0° in the multipoint case. Furthermore, while the total drag was reduced by 53% in
Table 6. Design points and weighting for multipoint optimization

<table>
<thead>
<tr>
<th>Design Pt</th>
<th>Quadrature Weight (T)</th>
<th>Aircraft Weight (W) [lbs]</th>
<th>Mach No. (M)</th>
<th>Reynolds No. (Re)</th>
<th>Lift Coefficient (C_l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>63757</td>
<td>0.48</td>
<td>$12.5 \times 10^6$</td>
<td>0.68</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>63757</td>
<td>0.54</td>
<td>$14.1 \times 10^6$</td>
<td>0.54</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>63757</td>
<td>0.60</td>
<td>$15.7 \times 10^6$</td>
<td>0.44</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>60754</td>
<td>0.48</td>
<td>$12.5 \times 10^6$</td>
<td>0.65</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>60754</td>
<td>0.54</td>
<td>$14.1 \times 10^6$</td>
<td>0.51</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>60754</td>
<td>0.60</td>
<td>$15.7 \times 10^6$</td>
<td>0.42</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>57751</td>
<td>0.48</td>
<td>$12.5 \times 10^6$</td>
<td>0.62</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>57751</td>
<td>0.54</td>
<td>$14.1 \times 10^6$</td>
<td>0.49</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>57751</td>
<td>0.60</td>
<td>$15.7 \times 10^6$</td>
<td>0.40</td>
</tr>
</tbody>
</table>

the single-point optimization, it was reduced by 50% in the multipoint optimization. This illustrates that the added robustness in the design (now optimized over a range of conditions) incurs a penalty in the on-design performance. It also exemplifies the importance of the designer’s role in carefully selecting and weighting the design points appropriately. For example, if the Q400 normally cruises at the high-speed Mach number of 0.60, then the designer might choose to place more importance on those operating points.

V. Conclusions

A two-dimensional RANS solver making use of the Spalart-Allmaras turbulence model has been extended to incorporate an iterative laminar-turbulent transition prediction methodology. With reasonable grid density, the boundary-layer properties can be computed directly from the Navier-Stokes solution with sufficient accuracy. The compressible form of the AHD criterion and the simplified $e^N$ envelope method have been implemented, verified, and validated by comparison to numerical and experimental data.

The RANS solver was subsequently employed in a gradient-based sequential quadratic programming shape optimization framework using the SNOPT optimization suite. The gradients are evaluated using a new augmented discrete-adjoint formulation for transition prediction in a RANS solver, the accuracy of which has been verified. The resulting optimization framework has been applied to the design of natural-laminar-flow airfoils through single and multipoint optimization. Such applications demonstrate the efficacy and practicality of using high-fidelity aerodynamic shape optimization as an NLF design tool in the subsonic and transonic flight regime. Future work will consider Pareto front studies which aid the design of airfoils that exploit laminar flow under ideal conditions, but also perform well when transition occurs further forward than expected. Future work will also consider the extension of the current methodology to three dimensions, incorporating a crossflow transition criterion.

VI. Acknowledgements

The authors gratefully acknowledge the financial assistance of the Ontario Government, Mathematics of Information Technology and Complex Systems, Canada Research Chairs Program, Bombardier Aerospace, and the University of Toronto.
Table 7. Summary of multipoint optimization results

<table>
<thead>
<tr>
<th>Design Pt.</th>
<th>$C_d$</th>
<th>$C_{dp}$</th>
<th>$C_{dv}$</th>
<th>$C_l$</th>
<th>$C_m$</th>
<th>$T_{up}(x/c)$</th>
<th>$T_{lo}(x/c)$</th>
<th>AoA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Initial</td>
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<td>0.00345</td>
<td>0.00458</td>
<td>0.6795</td>
<td>-0.07315</td>
<td>0.0172</td>
<td>0.5393</td>
<td>3.5050°</td>
</tr>
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<td>0.00241</td>
<td>0.6799</td>
<td>-0.08278</td>
<td>0.5509</td>
<td>0.7767</td>
<td>2.5651°</td>
</tr>
<tr>
<td>2 Initial</td>
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<td>0.00253</td>
<td>0.00439</td>
<td>0.5400</td>
<td>-0.07691</td>
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<td>2.2177°</td>
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<tr>
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<td>0.5399</td>
<td>-0.08434</td>
<td>0.6656</td>
<td>0.7583</td>
<td>1.2498°</td>
</tr>
<tr>
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<td>0.00613</td>
<td>0.00205</td>
<td>0.00408</td>
<td>0.4400</td>
<td>-0.08123</td>
<td>0.1328</td>
<td>0.4932</td>
<td>1.2852°</td>
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<td>0.00105</td>
<td>0.00199</td>
<td>0.4400</td>
<td>-0.08730</td>
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<td>0.3184°</td>
</tr>
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<td>0.00323</td>
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<td>-0.07328</td>
<td>0.0199</td>
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<td>3.2717°</td>
</tr>
<tr>
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<td>0.6194</td>
<td>0.7741</td>
<td>2.3059°</td>
</tr>
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<td>0.00234</td>
<td>0.00436</td>
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<td>-0.07704</td>
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<td>0.5103</td>
<td>1.9920°</td>
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<tr>
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<td>0.5100</td>
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<td>0.7558</td>
<td>1.0383°</td>
</tr>
<tr>
<td>6 Initial</td>
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<td>0.00194</td>
<td>0.00405</td>
<td>0.4201</td>
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<td>1.1430°</td>
</tr>
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<td>0.00200</td>
<td>0.4202</td>
<td>-0.08662</td>
<td>0.6683</td>
<td>0.7226</td>
<td>0.1875°</td>
</tr>
<tr>
<td>7 Initial</td>
<td>0.00761</td>
<td>0.00298</td>
<td>0.00463</td>
<td>0.6200</td>
<td>-0.07355</td>
<td>0.0232</td>
<td>0.5321</td>
<td>3.0357°</td>
</tr>
<tr>
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<td>0.00155</td>
<td>0.00213</td>
<td>0.6201</td>
<td>-0.08191</td>
<td>0.6421</td>
<td>0.7717</td>
<td>2.0713°</td>
</tr>
<tr>
<td>8 Initial</td>
<td>0.00656</td>
<td>0.00222</td>
<td>0.00434</td>
<td>0.4900</td>
<td>-0.07710</td>
<td>0.0860</td>
<td>0.5082</td>
<td>1.8419°</td>
</tr>
<tr>
<td>Optimized</td>
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<td>0.00111</td>
<td>0.00200</td>
<td>0.4899</td>
<td>-0.08287</td>
<td>0.6705</td>
<td>0.7543</td>
<td>0.8963°</td>
</tr>
<tr>
<td>9 Initial</td>
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<td>0.00401</td>
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<td>-0.08133</td>
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<td>-0.08590</td>
<td>0.6701</td>
<td>0.6977</td>
<td>0.0527°</td>
</tr>
</tbody>
</table>

References

Figure 4. Multipoint Optimization Results


